Numerical Solution of Fractional Differential Equation By Using Laplace Transform Method and Applications

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Abstract

In this paper, we develop Laplace transform method to solve fractional differential equations with Liouville fractional derivatives. Also we see some solved examples numerically, which will show the efficiency and convenience of Laplace Transform Method.

Keywords: Mittag-Leffler fuction, Riemann-Liouville fractional derivative, Laplace Transform method.

1 Introduction

Fractional calculus has a long history, mentioned by Gottfried Withlem Leibniz in 1695. Leibniz's note to led to the the factional calculus, which was developed by Liouville, Grunwald, Letnikov and Riemann. In 19^{th} century it is developed by Goldman (1949), Starkey (1954), R.P. Agarwal (1953), Scott (1955), Mikuniski (1959) , Holbrook (1966), Oldham and Spanier (1974), Miller and Ross (1993), K.Nishimoto (1987), S.C.Dutta Roy (1967), Mainaradi (1991), L.Devnath (1992), Kolwanker and Gangal (1994), Oustaloup (1994), Tom Hartley (1998), Igor Podlubny (1999), R.K.Saxena (2002), A.A.Kilbas, H.M.Srivastava and J.J.Trujillo (2006), Shantanu Das (2008) , Lakshmikantham (2009), F.Mainardi (2010), Hilfer (2011) and several others are notable. There have been many excellent books and monographs on this field [14,19 ,21,22, 25,30,31].

In fact many scientific areas are currently paying attention to the factional calculus concepts and we can refer its adoption in viscoelasticity and damping diffusing and wave propogation are electromagnetism chaos and fractals heat transfer, biology electronics signal processing, robotics, system identification traffic systems genetic algorithms percolation, modeling and identification, telecommunications modeling and identification, telecommunications chemistry, irreversibility, physics control system as well as economy and finance.

The Laplace transform or Laplace transformation is a method for solving linear differential equations arising in physics and Engineering [18]. Denoted $L[\varphi(t)]$, it is a linear operator of a function $\varphi(t)$ with a real argument t ($t \geq 0$) that transforms it to a function $\varphi(s)$ with a complex arguments.

The Laplace transform of a function $\varphi(t)$ of a real variable $t \in \mathbb{R}^+ = (0,\infty)$ is defined as

$$(L\varphi)(s) = L[\varphi(t)](s) = \varphi(s) = \int_0^\infty e^{-st}\varphi(t)dt \quad (s \in \mathbb{C}).$$

The inverse Laplace transform is given for $x \in \mathbb{R}^+$ by the formula

$$(L^{-1}g)(x) = L^{-1}[g(s)](x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx}g(s)ds \qquad (\gamma = \Re(s) > \sigma\varphi).$$

Where $\Re$ stands for real part of complex number z. The Laplace Transform method is based on the relation

$$L[(D_{0+}^{\alpha}y)(x)](s) = s^{\alpha}[Ly](s) - \sum_{k=0}^{n-1} s^{n-1-k}D^k(I_{0+}^{n-\alpha}y)(0+) \qquad (\Re(s) > q_0)$$

and in accordance with Riemann-Liouville fractional derivative, is equivalent to the following one

$$L[(D_{0+}^{\alpha}y)(x)](s) = s^{\alpha}[Ly](s) - \sum_{j=1}^{l} d_j s^{j-1} \quad (l-1 < \alpha \leqq l; l \in \mathbb{N})$$

$$d_j = (D_{0+}^{\alpha-j}y)(0+) \qquad (j = 1,\cdots,l).$$

The paper is organized as follows: Section 2 contains Preliminary definitions from fractional calculus. In section 3, we develop Laplace Transform method for fractional differential equation with constant coefficient. Also we see one theorem. In section 4 , we see illustrated examples. Last section includes conclusion.

2 Basic Definitions:

In this section, we consider the following definitions of fractional derivatives which are useful for furthur developments.

The Riemann-Lioville fractional derivatives D_{a+}^{α} y and $D_{b-}^{\alpha}y$ of order $\alpha \in \mathbb{C}(\Re(\alpha) \geqq 0)$ are defined by

$$\begin{aligned}(D_{a+}^{\alpha}y) &= \left(\frac{d}{dx}\right)^n (I_{a+}^{n-\alpha}y)(x) \\ &= \frac{1}{(n-\alpha)}\left(\frac{d}{dx}\right)\int_a^x \frac{y(t)dt}{(x-t)^{\alpha-n+1}} \qquad (n = [\Re(\alpha)] + 1; x > a)\end{aligned}$$

$$(D_{b-}^{\alpha}y) = \left(-\frac{d}{dx}\right)^{n}(I_{b-}^{n-\alpha}y)(x)$$
$$= \frac{1}{(n-\alpha)}\left(-\frac{d}{dx}\right)\int_x^b \frac{y(t)dt}{(t-x)^{\alpha-n+1}} \qquad (n = [\Re(\alpha)] + 1;\, x < b)$$

respectively, where $[\Re(\alpha)]$ means the integral part of $\Re(\alpha)$.

Mittage-Leffer function:-

The Mittage-Leffler function plays a very important role in the theory of integer order differential equation [1]. More detailed information may be found in the books by Erdelyi et al. ([1], vol.3, section 18.1). The one parameter generalisation of the exponential function denoted by $E\alpha(z)$ and introduced by Mittag-Leffer is defined as

$$E\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \qquad (z \in \mathbb{C};\, \Re(\alpha) > 0)$$

was introduced by Mittage-Leffer and is so known as the Mittag-Leffer function. It is an entire function of z with order $[\Re(\alpha)]^{-1}$ and type one.

Mittag-Leffler function in two parameters:

The two parameter Mittage-Leffer type function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \qquad (z, \beta \in \mathbb{C};\Re(\alpha) > 0)$$

Laplace Transform of Mittag-Leffler function:

$$L[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})](s) = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}\,(\Re(s) > 0;\, \lambda \in \mathbb{C};\, |\lambda s^{-\alpha}| < 1).$$

3 Laplace Transform Method for fractional nonhomogeneous Differential equation with constant coefficient:

In this section, we consider a scheme for solving the one dimensional fractional nonhomogeneous differential equation with constant coefficients of the form

$$\sum_{k=1}^{m} A_k(D_{0+}^{\alpha k}y)(x) + A_0 y(x) = f(x)(x > 0) \qquad (3.1)$$

with $m \in \mathbb{N}$; $0 < \Re(\alpha_1) < \cdots < \Re(\alpha_m)$; $A_o, A_1 \cdots A_m \in \mathbb{R}$ and involving the Liouville fractional derivative $(D_{0+}^{\alpha k}y)(k = 1 \cdots m)$

If $n = [\Re(\alpha)] + 1, y(x) \in AC^m[0, b]$ for any $b > 0$ and

$$|y(x)| \leqq Be^{q_0 x} \qquad (x > b > 0)$$

holds for constants $B > 0$ and $q_0 > 0$, and if $y^k(0) = 0 (k = 0,1,\cdots,n-1)$, then the relation

$$L[D_{0+}^{\alpha}y)(x)](s) = s^{\alpha}[Ly](s) \tag{3.2}$$

is valid for $\Re(s) > q_0$. Applying the Laplace transform to equation (3.1) and using (3.2), we get

$$L\left[\sum_{k=1}^{m} A_k(D_{0+}^{\alpha k}y)(x) + A_0 y(x)\right](s) = L[f(x)](s)$$

$$\sum_{k=1}^{m} L[A_k(D_{0+}^{\alpha k}y)(x)](s) + L[A_0 y(x)](s) = L[f(x)](s)$$

$$\sum_{k=1}^{m} A_k s^{\alpha k}[Ly](s) + A_0[Ly](s) = L[f(x)](s)$$

$$\left[A_0 + \sum_{k=1}^{m} A_k s^{\alpha k}\right][Ly](s) = L[f(x)](s) \tag{3.3}$$

Taking inverse Laplace transform on both side, we get a particular solution to the equation (3.1) in the form

$$y(x) = \left(L^{-1}\left[\frac{[Lf](s)}{A_0 + \sum_{k=1}^{m} A_k s^{\alpha k}}\right]\right)(x) \tag{3.4}$$

Now we see one theorem.

Theorem: Let $l - 1 < \alpha \leqq l$ ($l \in \mathbb{N}$) and $\lambda \in \mathbb{R}$. Then the functions

$$y_j(x) = x^{\alpha-j} E_{\alpha,\alpha+1-j}(\lambda x^{\alpha}) (j = 1,\cdots l) \tag{3.5}$$

yield the fundamental system of solutions to equation

$$(D_{0+}^{\alpha}y)(x) - \lambda y(x) = 0 \quad (x > 0; l - 1 < \alpha \leqq l; l \in \mathbb{N}; \lambda \in \mathbb{R}).$$

Proof:- We have

$$(D_{0+}^{\alpha}y)(x) - \lambda y(x) = 0$$

Taking Laplace transform on both side, we get

$$L[(D_{0+}^{\alpha}y)(x)](s) - \lambda L[y(x)](s) = 0$$

But,

$$L[(D^{\alpha}_{0+}y)(x)](s) = s^{\alpha}[Ly](s) - \sum_{j=1}^{l} d_j s^{j-1} \qquad (l-1 < \alpha \leqq l \in \mathbb{N})$$

$$\therefore s^{\alpha}[Ly](s) - \sum_{j=1}^{l} d_j s^{j-1} - \lambda[Ly](s) = 0$$

$$[Ly](s) = \sum_{j=1}^{l} d_j \frac{s^{j-1}}{s^{\alpha} - \lambda} \tag{3.6}$$

We have

$$L[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})](s) = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}$$

put $\beta = \alpha + 1 - j$ in above equation, we get

$$L[t^{\alpha-j}E_{\alpha,\alpha+1-j}(\lambda t^{\alpha})](s) = \frac{s^{j-1}}{s^{\alpha} - \lambda}$$

from equation (3.6) and taking inverse Laplace transform on both side, we get

$$y(x) = \sum_{j=1}^{l} d_j y_j(x), \qquad y_j(x) = x^{\alpha-j} E_{\alpha,\alpha+1-j}$$

This is the proof.

4 Numerical Examples:

Example 4.1:- The equation

$$(D^{l-1/2}_{0+}y)(x) - \lambda y(x) = 0 \quad (x > 0; l \in \mathbb{N}, \lambda \in \mathbb{R}),$$

has the fundamental system of solutions given by

$$y_j(x) = x^{l-j-1/2}E_{l-\frac{1}{2},l-j+1/2}(\lambda x^{l-1/2}) \qquad (j = 1,\cdots l)$$

solution:- The given equation is

$$(D^{l-1/2}_{0+}y)(x) - \lambda y(x) = 0$$

Taking Laplace transform on both side, we get

$$L[(D^{l-1/2}_{0+}y)(x)](s) = \lambda L[y(x)](s) \tag{4.1}$$

But we know that

$$L[(D^{\alpha}_{0+}y)(x)](s) = s^{\alpha}[Ly](s) - \sum_{j=1}^{m} d_j s^{j-1} \qquad (m - 1 < \alpha \leqq m; m \in \mathbb{N})$$

Putting $\alpha = l - 1/2$

$$L[(D_{0+}^{l-1/2}y)(x)](s) = s^{l-1/2}[Ly](s) - \sum_{j=1}^{m} d_j s^{j-1} \tag{4.2}$$

from equation (4.1) and (4.2)

$$s^{l-1/2}[Ly](s) - \lambda[Ly](s) = \sum_{j=1}^{m} d_j\, s^{j-1}$$

$$[Ly](s)) = \sum_{j=1}^{m} dj \frac{s^{j-1}}{s^{l-1/2} - \lambda} \tag{4.3}$$

We know that

$$L[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})](s) = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}$$

Put $\alpha = l - 1/2$ and $\beta = l - j + ½$

$$L[t^{l-j-1/2}E_{l-\frac{1}{2},l-j+1/2}(\lambda t^{l-1/2})](s) = \frac{s^{j-1}}{s^{l-1/2}-\lambda}$$

from equation (4.3) and taking inverse Laplace transform on both side, we get

$$y(x) = \sum_{j=1}^{m} d_j x^{l-j-1/2} E_{l-1/2,l-j+1/2}(\lambda x^{l-1/2}), \qquad y(x) = \sum_{j=1}^{m} d_j y_j(x)$$

$$y_j(x) = x^{l-j-1/2}E_{l-1/2,l-j+1/2}(\lambda x^{l-1/2})$$

Example 4.2:- The following ordinary differential of order $l \in \mathbb{N}$

$$y^{l}(x) - \lambda y(x) = 0 \qquad (x > 0; l \in \mathbb{N})$$

has the fundamental system of solution given by

$$y_j(x) = x^{l-j}E_{l,l+1-j}(\lambda x^{l}) \qquad (j = 1 \cdots, l).$$

Solution:- Given that

$$y^{(l)}(x) - \lambda y(x) = 0$$

$$\text{i.e.} \qquad (D_{0+}{}^{l}y)(x) - \lambda y(x) = 0$$

Taking Laplace Transform on both side, we get

$$L[(D_{0+}^{l}y)(x)](s) - \lambda L[y(x)](s) = 0 \quad (4.4)$$

But we know

$$L[(D_{0+}^{\alpha}y)(x)](s) = s^{\alpha}[Ly](s) - \sum_{j=1}^{m} d_j s^{j-1}$$

Put $\alpha = l$

$$L[(D_{0+}^{l}y)(x)](s) = s^{l}[Ly](s) - \sum_{j=1}^{m} d_j s^{j-1}$$

So equation (4.4) becomes

$$s^{l}[Ly](s) - \sum_{j=1}^{m} d_j s^{j-1} = \lambda[Ly](s)$$

$$[Ly](s) = \sum_{j=1}^{m} d_j \frac{s^{j-1}}{s^{l} - \lambda} \quad (4.5)$$

We have

$$\mathrm{L}[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})](\mathrm{s}) = \frac{s^{\alpha-\beta}}{s^{\alpha-\lambda}}$$

Put $\alpha = l$, and $\beta = l + 1 - j$

$$L[t^{l-j}E_{l,l+1-j}(\lambda t^{l})](s) = \frac{s^{j-1}}{s^{l} - \lambda}$$

from equation (4.5) and taking inverse Laplace transform on both side, we get

$$y(x) = \sum_{j=1}^{m} d_j x^{l-j} E_{l,l+1-j}(\lambda x^{l}), \qquad y(x) = \sum_{j=1}^{m} d_j y_j(x)$$

$$y_j(x) = x^{l-j}{}_{-j\,E_{l,l+1-j}}(\lambda x^{l})$$

5 Conclusion:

Laplace transformation is powerful tool using in different areas of mathematics physics and engineering. In this study, we gave some details of the laplace Transform. In this paper, we developed Laplace Transform method for solving fractional differential equation with constant coefficient. This method is very beneficial and easy to calculate accurate solutions of the given examples. Due to this efficiency to the Laplace transformable equations the Laplace transform equation made a good advancement in the research field.

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