On Some Characterizations of $CM'(2,2)$ Near Ring

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ABSTRACT

In this paper we introduce a new notion called $CM'(2,2)$ near ring. Also, $CM'(2,2)$ near ring has been generalised to a new notion called Strong $CM'(2,2)$ near ring. The properties of $CM'(2,2)$ near ring and Strong $CM'(2,2)$ near ring is discussed using the concepts like zero symmetric, simple, near field and α_1 near ring. We have also characterized that any homomorphic image of a $CM'(2,2)$ near ring is again a $CM'(2,2)$ near ring. It is shown that every Strong $CM'(2,2)$ near ring is a $CM'(2,2)$ near ring. We have also proved that a zero symmetric Strong $CM'(2,2)$ near ring with α_1 near ring is reduced.

Keywords: -

Boolean, Commutative, Ideal, Near field, Zero Symmetric.

1.Introduction

Near rings can be thought of as generalized rings: if in a ring we ignore the commutativity of addition and one distributive law, we get a near ring. Gunter Pilz "Near Rings" is an extensive collection of the work done in the area of near rings.

Throughout this paper N stands for a right near ring $(N, +, .)$, with at least two elements and '0' denotes the identity element of the group $(N, +)$ and we write xy for $x.y$ for any two elements x, y of N. Obviously $0n = 0$ for all $n \in N$. If, in addition, $n0 = 0$ for all $n \in N$ then we say that N is **zero symmetric.** For any subset A of N, we denote A^* the set of all non-zero elements of A. In particular $N^* = N - \{0\}$.

2.Preliminaries

Definition 2.1 [6]

A **right near ring** is a non-empty set N together with two binary operations "+" and "." such that (i) $(N,+)$ is a group (ii) $(N,.)$ is a semigroup (iii) $(n_1+n_2)n_3 = n_1n_3 + n_2n_3$ for all $n_1, n_2, n_3 \in N$.

Definition 2.2 [6]

If $(N,.)$ is commutative we call N itself a **commutative near ring**.

Definition 2.3 [6]

Let $N, N' \in N$. Let $h: N \longrightarrow N'$ is called a **near-ring homomorphism** if for all $m, n \in N$ (i) $h(m+n) = h(m) + h(n)$ (ii) $h(mn) = h(m)h(n)$.

Definition 2.4 [6]

N is called **subdirectly irreducible** if N is not isomorphic to a non-trivial subdirect product of near rings.

Result 2.5 [6]

Every near ring is isomorphic to a sub direct product of sub directly irreducible near rings.

Definition 2.6 [6]

A non-empty subset I of N is called (i) a **left ideal** of N if $(I,+)$ is a normal subgroup of $(N,+)$ and $n(n'+i) - nn' \in N$ and $i \in I$. (ii) a **right ideal** of N if $(I,+)$ is a normal subgroup of $(N,+)$ and if $IN \subseteq I$. (iii) an **ideal** of N if I is both a left ideal and right ideal of N.

Clearly $\{0\}$ and N are ideals of N. These are called trivial ideals.

Definition 2.7 [7]

N is called a **near field** if it contains an identity and each non zero element has a multiplicative inverse.

Definition 2.8 [6]

A near ring N is called **Boolean** if and only if $x^2 = x$ for all $x \in N$.

Definition 2.9 [6]

N is **simple** if and only if N has no non-trivial ideals.

Definition 2.10 [9]

Let N be a right near ring. If for every $a \in N$ there exists $x \in N$ such that $a = xax$ then we say N is an $\boldsymbol{\alpha_1}$ **near ring**.

Result 2.11 [5]

Every simple near ring is sub directly irreducible.

Definition 2.12 [6]

A subgroup M of N is called an ***N*-subgroup** if $NM \subset M$ and an **invariant *N*-subgroup** if, in addition, $MN \subset M$.

Definition 2.13 [7]

An additive group A of N is called a **left *N*-subgroup (right N-subgroup)** if $NA \subseteq A(AN \subseteq A)$ Where $NA = \{ra|r \in N, a \in A\}$ $(AN = \{ar|r \in N, a \in A\})$.

Result 2.14 [1]

N is Zero symmetric if and only if every ideal of N is an N-subgroup of N.

Definition 2.15 [2]

An element $a \in N$ is said to be **nilpotent,** if $a^k = 0$ for some positive integer k.

Definition 2.16

N is called an **integral near ring** if N has no non-zero zero divisors.

Result 2.17 [8]

N has no nonzero nilpotent elements if and only if $a^2 = 0 \Longrightarrow a = 0$ for all $a \in N$.

Result 2.18 [6]

Let N be Zero Symmetric. Then the following are equivalent:

(i) N has no nonzero nilpotent elements.

(ii) N is a subdirect product of integral near rings.

Definition 2.19 [4]

A near ring N is said to be **reduced** if N has no non-zero nilpotent elements.

3.Main Results

3.1 CM′(2,2) Near Ring

Definition 3.1.1

A near ring N is said to be a **CM′(2,2) near ring** if for every $a \in N$, there exists $x \in N^*$ such that $xax = x^2a^2$.

Proposition 3.1.2

Let N be a CM′(2,2) near ring and if $xa = 0$ then $x^2a^2 = 0$.

Proof:

Let $a \in N$. Since N is a $CM'(2,2)$ near ring, there exists $x \in N^*$ such that $xax = x^2a^2$. If $xa = 0$ we get $x^2a^2 = 0x = 0$. Thus $x^2a^2 = 0$.

Theorem 3.1.3

Let N be a commutative $CM'(2,2)$ near ring. For all positive integer $k > 1$ and for all $a \in N$, N is $CM'(2,2)$ near ring such that $b^kab^k = b^{2k}a^2$.

Proof:

Since N is $CM'(2,2)$ near ring, $bab = b^2a^2$. If $k = 1$, the result is obvious. If $k = 2$, $b^2ab^2 = b(bab)b = b(b^2a^2)b = b^3(a^2b) = b^3(ba^2) = b^4a^2$. That is $b^2ab^2 = b^4a^2$. Continuing this process, we get $b^kab^k = b^{2k}a^2$ for all positive integers $k > 1$.

Proposition 3.1.4

Any homomorphic image of a CM′(2,2) near ring is a CM′(2,2) near ring.

Proof:

Let N be a $CM'(2,2)$ near ring. Let $f: N \longrightarrow N'$ be a homomorphism. Since N is a $CM'(2,2)$ near ring it demands that for every $a \in N$, there exists $b \in N^*$ such that $bab = b^2a^2$. Let $x', y' \in N'$, then there exists $x, y \in N$ such that $f(x) = x'$ and $f(y) = y'$. Now $y'x'y' = f(y)f(x)f(y) = f(yxy) = f(y^2x^2) = f(y^2)f(x^2) = [f(y)]^2[f(x)]^2 = (y')^2(x')^2$. That is $y'x'y' = (y')^2(x')^2$.

Theorem 3.1.5

Every $CM'(2,2)$ near ring is isomorphic to a sub direct product of sub directly irreducible $CM'(2,2)$ near ring.

Proof:

Let N be a $CM'(2,2)$ near ring. From the Result 2.4, we get N is isomorphic to a sub direct product of sub directly irreducible near rings ($N_i's$) (say) and each N_i is a homomorphic image of N under projection map π_i. The desired result now follows from Proposition 3.1.4.

Proposition 3.1.6

If I is an ideal of $CM'(2,2)$ near ring N then ${}^{N}/_{I}$ is also a $CM'(2,2)$ near ring.

Proof:

The function $\varphi: N \longrightarrow {}^{N}/_{I}$ be defined by $\varphi(x) = I + x$ where $x \in N$ is an epimorphism. The rest of the proof is taken care of by Proposition 3.1.4.

Theorem 3.1.7

Let N be a near ring. Each of the following statements implies that N is a $CM'(2,2)$ near ring. (i) N has identity '1'. (ii) N is a near field.

Proof:

(i) Follows by taking $a = 1$ in the definition of $CM'(2,2)$ near ring.

(ii) Follow from (i).

3.2 Strong CM′(2,2) Near Ring

Definition 3.2.1

A $CM'(2,2)$ near ring is said to be Strong $CM'(2,2)$ near ring if $bab = b^2a^2$ for all $a, b \in N$.

Theorem 3.2.2

Every Strong $CM'(2,2)$ near ring is a $CM'(2,2)$ near ring.

Proof:

Straight Forward.

Proposition 3.2.3

Let N be a Strong $CM'(2,2)$ near ring. If N is Boolean, then $bab = ba$ for all $a, b \in N$.

Proof:

Since N is Strong $CM'(2,2)$ near ring $bab = b^2a^2$ for all $a, b \in N$. Since N is Boolean $a^2 = a, b^2 = b$ for all $a, b \in N$. Now $bab = b^2a^2 = ba$.

Proposition 3.2.4

Any homomorphic image of a Strong CM′(2,2) near ring is a Strong $CM'(2,2)$ near ring.

Proof:

Let N be a Strong $CM'(2,2)$ near ring and let $f: N \longrightarrow N'$ be a homomorphism. Since N is a Strong $CM'(2,2)$ near ring it demands that $bab = b^2a^2$ for all $a, b \in N$. Let $x', y' \in N'$ then there exists $x, y \in N$ such that $f(x) = x'$ and $f(y) = y'$. Clearly, then $y'x'y' = (y')^2(x')^2$ and the desired result now follows.

Theorem 3.2.5

Every Strong $CM'(2,2)$ near ring is isomorphic to a sub direct product of sub directly irreducible Strong $CM'(2,2)$ near ring.

Proof:

Let N be a Strong $CM'(2,2)$ near ring. From the Result 2.4, we get N is isomorphic to a sub direct product of sub directly irreducible near rings $N_i's$ (say) and each N_i is a homomorphic image of N under projection map π_i. The desired result now follows from Proposition 3.2.4.

Proposition 3.2.6

If I is an ideal of Strong $CM'(2,2)$ near ring N then $N/_I$ is also a Strong $CM'(2,2)$ near ring.

Proof:

The function $\varphi: N \longrightarrow N/_I$ be defined by $\varphi(x) = I + x$ where $x \in N$ is an epimorphism. The rest of the proof is taken care by the Proposition 3.2.4.

Theorem 3.2.7

Let N be a Strong $CM'(2,2)$ near ring. If N is α_1 near ring then (i) N is simple. (ii) N is sub directly irreducible.

Proof:

(i) Suppose N has a non-trivial ideal I. Let b be a nonzero element of I. Now let $a \in N$. Since N is a Strong $CM'(2,2)$ near ring, by definition $bab = b^2a^2$. Given that N is α_1 near ring $\Longrightarrow bab = a$ where a is arbitrary. Now $a = b^2a^2 \Longrightarrow a \in IN \subset I \Longrightarrow a \in I$. Therefore $N \subset I$ and hence $I = N$. Thus N is simple.

(ii) Follows from (i) and from Result 2.7

Theorem 3.2.8

If N is Zero symmetric Strong $CM'(2,2)$ near ring then every ideal of N is an invariant N-subgroup of N.

Proof:

Let I be an ideal of N. Since N is Zero symmetric and hence by Result 2.8. Hence I becomes an N-subgroup of N. That is $NI \subset I$. Obviously $IN \subset I$. Thus I becomes an invariant N-subgroup.

Theorem 3.2.9

Let N be a Strong $CM'(2,2)$ near ring. If N is α_1 near ring then for every $a \in N$, there exists $x \in N^*$ such that $a = x^2a^2$.

Proof:

Since N is α_1 near ring for every $a \in N$, there exists $x \in N^*$ such that $a = xax$.... (1). Since N is a Strong $CM'(2,2)$ near ring, $xax = x^2a^2$.... (2). From (1) and (2) we get $a = x^2a^2$.

Theorem 3.2.10

Let N be a Zero symmetric Strong $CM'(2,2)$ near ring. If N is α_1 near ring then N is reduced.

Proof:

Let $a \in N$. Suppose $a^2 = 0$. Now by Theorem 3.2.9 there exists $x \in N^*$ such that $a = x^2a^2 = x^2 0 = 0$ (Since $N = N_0$). Hence $a = 0$. Now by Result 2.9, N has no nonzero nilpotent elements.

Corollary 3.2.11

Let N be a Zero symmetric Strong $CM'(2,2)$ near ring. If N is α_1 near ring then N is subdirect product of integral near rings.

Proof:

Let N be a Strong $CM'(2,2)$ near ring. Since N is α_1 near ring, by Theorem 3.2.10, N is reduced. As N is Zero Symmetric, the desired result now follows from Result 2.10

4.References

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