The Spectrum of the Q-vertex join and Q-edge join of two graphs

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Abstract

The Q-graph $\mathcal{Q}(H)$ of a graph H is the graph obtained from H by inserting a new vertex into every edge of H and by joining by edges those pairs of these new vertices which lie on adjacent edges of H. In this paper, we determine the Adjacency, Laplacian and Signless Laplacian spectra of Q-vertex join and Q-edge join (denoted by $H_1 \vee_{\boxdot} H_2$ and $H_1 \vee_{\boxminus} H_2$ respectively) of connected regular graph H_1 with an arbitrary graph H_2 in terms of their eigenvalues. As the application, we calculate the number of spanning trees of $H_1 \vee_{\boxdot} H_2$ and $H_1 \vee_{\boxminus} H_2$ of connected regular graph H_1 with an arbitrary graph H_2. Also we construct infinitely many pairs of $A-$ cospectral mates, $L-$cospectral mates and $Q-$cospectral mates.

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1 Introduction

Let $H = (V(H), E(H))$ be the graph with vertex set $V(H) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(H)$. The adjacency matrix of H, denoted by $A(H) = [a_{ij}]$ is an $n \times n$ symmetric matrix such that $a_{ij} = 1$ if vertices v_i and v_j are adjacent and 0 otherwise. We denote $d_i = d_H(v_i)$ by the degree of vertex v_i in H. If $D(H) = diag(d_1, d_2, \dots, d_n)$ is the diagonal matrix of vertex degree in H, then we define the *Laplacian matrix* $L(H)$ and *Signless Laplacian matrix* $Q(H)$ as $L(H) = D(H) - A(H)$ and $Q(H) = D(H) + A(H)$ respectively.

For a given matrix M of size n, we denote the characteristic polynomial $det(xI_n - M)$ by

$$\phi(M; x) = det(xI_n - M).$$

where I_n is the identity matrix of size n.
For a graph H, $\phi(A(H))$, $\phi(L(H))$, $\phi(Q(H))$ are respectively called *Adjacency*, *Laplacian* and *Signless Laplacian characteristic polynomial* of H and its roots are *Adjacency*, *Laplacian* and *Signless Laplacian* eigen values of H respectively. The Adjacency eigen value of H is denoted by $\theta_1(H) \geq \theta_2(H) \geq \cdots \theta_n(H)$ are called the A- *spectrum* of H. Similarly the Laplacian eigen value and Signless Laplacain eigen value of H are denoted by $0 = \mu_1(H) \leq \mu_2(H) \leq \cdots \leq \mu_n(H)$ and $\eta_1(H) \leq \eta_2(H) \leq \cdots \leq \eta_n(H)$ and are called *L-spectrum* and *Q-spectrum* respectively.

Many graph operations such as the disjoint union, join, the corona, the edge corona, the neighborhood corona etc.have been introduced till now and their spectra are also computed. The join [10] of two graphs is their disjoint union together with all edges that connect all the vertices of the first graph with all the vertices of the second graph. In [6, 7] two new joins namely subdivision vertex join and subdivision edge join are introduced and the spectral properties are investigated. In [8] the adjacency, Laplacian and signless Laplacian spectra of R-edge join and R-vertex join of a connected regular graph with an arbitrary regular graph in terms of their eigen values are determined. The

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Q-graph [5] $\mathcal{Q}(H)$ of H is the graph obtained from H by inserting a new vertex into every edge of H and by joining those pairs of these new vertices by edges which lie on adjacent edges of H. The set of newly added vertices is denoted by $I(H)$.
In this paper we define two joins such as the Q-vertex join and the Q-edge join of two vertex disjoint graphs H_1 and H_2 denoted by $H_1 \vee_{\boxdot} H_2$ and $H_1 \vee_{\boxminus} H_2$ respectively. Also we determine the Adjacency, Laplacian and Signless Laplacian spectra of Q-vertex join and Q-edge join.

Definition 1.1. *The Q-vertex join of two vertex disjoint graphs H_1 and H_2, denoted by $H_1 \vee_{\boxdot} H_2$, is the graph obtained from $\mathcal{Q}(H_1)$ and H_2 by joining each vertex of $V(H_1)$ with every vertex of $V(H_2)$.*

Definition 1.2. *The Q-edge join of two vertex disjoint graphs H_1 and H_2, denoted by $H_1 \vee_{\boxminus} H_2$, is the graph obtained from $\mathcal{Q}(H_1)$ and H_2 by joining each vertex of $I(H_1)$ with every vertex of $V(H_2)$.*

Figure 1: $Q(C_3)$ and P_2

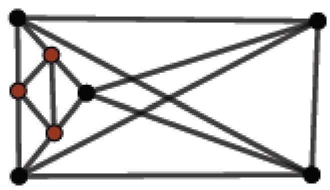

Figure 2: $C_3 \vee_{\boxdot} P_2$

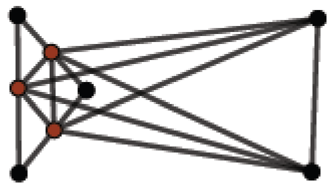

Figure 3: $C_3 \vee_{\boxminus} P_2$

All graphs considered in this paper are simple.

The following lemmas form the ground for the subsequent discussions.

Lemma 1.1. [1] *Let H be a graph with n vertices and m edges. The incidence matrix $R(H)$ of H is the $n \times m$ matrix with entry $r_{ij} = 1$ if the i^{th} vertex is incident to the j^{th} edge, and 0 otherwise. In particular if H is an r-regular graph then*

$$R(H)R(H)^T = A(H) + rI_n$$
$$R(H)^T R(H) = A(l(H)) + 2I_m$$

where $l(H)$ is the line graph of H.

Corollary 1.1. *Let H be an γ -regular graph with n vertices and m edges. For a constant a, we have*

$$(i)R(H)(\lambda I_m - aJ_{m\times m})^{-1}R(H)^T = \frac{\gamma I_n + A(H)}{\lambda} + \frac{a\gamma^2 J_{n\times n}}{\lambda(\lambda - ma)}$$
$$(ii)R(H)^T(\lambda I_m - aJ_{n\times n})^{-1}R(H) = \frac{2I_m + A(l(H))}{\lambda} + \frac{4aJ_{m\times m}}{\lambda(\lambda - na)}$$

where I_m is an identity matrix and $J_{n\times n}$ is a matrix of size $n \times n$ with all entries equal to one.

Lemma 1.2. [4] *Let M_1, M_2, M_3, M_4 be respectively $p\times p$, $p\times q$, $q\times p$, $q\times q$ matrices with M_1 and M_4 invertible. Then*

$$det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = det(M_4)\cdot det\left(M_1 - M_2{M_4}^{-1}M_3\right)$$
$$= det(M_1)\cdot det\left(M_4 - M_3{M_1}^{-1}M_2\right)$$

where $M_1 - M_2{M_4}^{-1}M_3$ and $M_4 - M_3{M_1}^{-1}M_2$ are called Schur complement of M_4 and M_1 respectively.

For a matrix M of order n we denoted 1_n and $J_{m\times n}$ the column vector of size n and matrix of size $m \times n$ with all the entries equal to one. M- coronal $\Gamma_M(\lambda)$ is defined to be the sum of entries of matrix $(\lambda I - M)^{-1}$ ie

$$\Gamma_M(\lambda) = 1_n^T(\lambda I - M)^{-1}1_n.$$

If M has constant row sum t, then $\Gamma_M(\lambda) = \frac{n}{\lambda - t}$.

Lemma 1.3. [5] *Let A be an $n \times n$ real matrix and adj(A) denote the adjugated matrix of A. Then*

$$det(A + aJ_{n\times n}) = det(A) + a1_n^T adj(A)1_n.$$

Moreover,

$$det(\lambda I_n - A - aJ_{n\times n}) = (1 - a\Gamma_A(\lambda))det(\lambda I_n - A).$$

2 Spectra of Q-vertex join of graphs

2.1 The Adjacency Spectra of $H_1 \vee_{\boxdot} H_2$

Let H_i be a graph on n_i vertices and m_i edges. Then the adjacency matrix of $H_1 \vee_{\boxdot} H_2$ can be written as:

$$A(H_1 \vee_{\boxdot} H_2) = \begin{bmatrix} 0 & R_1 & J_{n_1\times n_2} \\ R_1^T & A(l(H_1)) & 0 \\ J_{n_2\times n_1} & 0 & A(H_2) \end{bmatrix}$$

Theorem 2.1. *Let H_1 be an γ_1 regular graph on n_1 vertices and m_1 edges, and H_2 an arbitrary graph on n_2 vertices. Then the characteristic polynomial of $H_1 \vee_{\boxdot} H_2$ is*

$$\phi(H_1 \vee_{\boxdot} H_2; x) = \phi(H_2; x)\cdot(x+2)^{m_1 - n_1}(x^2 - x(2\gamma_1 - 2 + n_1\Gamma_{A(H_2)}(x)) - 2\gamma_1 + (2\gamma_1 - 2)\Gamma_{A(H_2)}(x))$$
$$\Pi_{i=1}^{n_1 - 1}(x^2 - x(\gamma_1 + \theta_i(H_1) - 2) - \gamma_1 - \theta_i(H_1))$$

Proof. The Characteristic polynomial of the adjacency matrix of $H_1 \vee_{\boxdot} H_2$ is given by

$$detA(H_1 \vee_{\boxdot} H_2) = \begin{vmatrix} xI_{n_1} & -R_1 & -J_{n_1\times n_2} \\ -R_1^T & xI_{m_1} - A(l(H_1)) & 0 \\ -J_{n_2\times n_1} & 0 & xI_{n_2} - A(H_2) \end{vmatrix}$$
$$= det(xI_{n_2} - A(H_2))\cdot det(S)$$
$$= \phi(H_2; x)\cdot det(S)$$

where

$$S = \begin{pmatrix} xI_{n_1} - \Gamma_{A(H_2)}(x)J_{n_1\times n_1} & -R_1 \\ -R_1^T & xI_{m_1} - A(l(H_1)) \end{pmatrix}$$

is the Schur complement of $xI_{n_2} - A(H_2)$.

$$
\begin{aligned}
det(S) &= det(xI_{n_1} - \Gamma_{A(H_2)}(x)J_{n_1\times n_1}) \cdot det\left(xI_{m_1} - A(l(H_1)) - R_1^T(xI_{n_1} - \Gamma_{A(H_2)}(x)J_{n_1\times n_1})^{-1}R_1\right) \\
&= (1-\Gamma_{A(H_2)}(x)\Gamma_0(x)) \cdot x^{n_1} \cdot det\left(xI_{m_1} - A(l(H_1)) - \frac{2I_{m_1} + A(l(H_1))}{x} + \frac{4\Gamma_{A(H_2)}(x)}{x(x-n_1\Gamma_{A(H_2)}(x))}J_{m_1\times m_1}\right) \\
&= (1-\Gamma_{A(H_2)}(x)\Gamma_0(x)) \cdot x^{n_1} \cdot \left(1 - \frac{4\Gamma_{A(H_2)}(x)}{x(x-n_1\Gamma_{A(H_2)}(x))}\Gamma_{A(l(H_1))+\frac{2I_{m_1}+A(l(H_1))}{x}}(x)\right) \cdot \\
&\quad det\left(xI_{m_1} - A(l(H_1)) - \frac{2I_{m_1} + A(l(H_1))}{x}\right)
\end{aligned}
$$

But $\Gamma_0(x) = \frac{n_1}{x}$ and $\Gamma_{A(l(G_1))+\frac{2I_{m_1}+A(l(H_1))}{x}(x)} = \frac{m_1 x}{x^2 - x(2\gamma_1 - 2) - 2\gamma_1}$

$$
\begin{aligned}
det(S) &= x^{n_1-1}\left(x - n_1\Gamma_{A(H_2)}(x)\right) \cdot \left(1 - \frac{4m_1\Gamma_{A(H_2)}(x)}{(x-n_1\Gamma_{A((H_2))}(x))(x^2 - x(2\gamma_1-2) - 2\gamma_1)}\right) \cdot \\
&\quad \Pi_{i=1}^{m_1}\left(x - \theta_i(l(H_1)) - \frac{2+\theta_i(l(H_1))}{x}\right) \\
&= (x+2)^{m_1-n_1}\left(x^2 - x\left(2\gamma_1 - 2 + n_1\Gamma_{A(H_2)}(x)\right) - 2\gamma_1 + (2\gamma_1-2)\Gamma_{A(H_2)}(x)\right) \cdot \\
&\quad \Pi_{i=1}^{n_1-1}\left(x^2 - x(\gamma_1 + \theta_i(H_1) - 2) - \gamma_1 - \theta_i(H_1)\right)
\end{aligned}
$$

Since H_1 is a regular graph with regularity $\gamma_1$and A-spectrum of $l(H_1)$ are $\theta_i(H_1)+\gamma_1-2$ for $i = 1,2,\cdots,n_1$ and -2 repeated $m_1 - n_1$ times.In the last step we use the fact that $\gamma_1 n_1 = 2m_1$ and $\theta_1(H_1) = \gamma_1$

Thus the characteristic polynomial of $H_1 \vee_{\boxdot} H_2$ is

$$
\begin{aligned}
\phi(H_1 \vee_{\boxdot} H_2; x) = \phi(H_2; x) \cdot (x+2)^{m_1-n_1}(x^2 - x(2\gamma_1 - 2 + n_1\Gamma_{A(H_2)}(x)) - 2\gamma_1 + (2\gamma_1 - 2)\Gamma_{A(H_2)}(x)) \\
\Pi_{i=1}^{n_1-1}(x^2 - x(\gamma_1 + \theta_i(H_1) - 2) - \gamma_1 - \theta_i(H_1))
\end{aligned}
$$

Since H_1 is a regular graph with regularity γ_1, in the last step we use the fact that $\gamma_1 n_1 = 2m_1$ and $\theta_1(H_1) = \gamma_1$. □

Corollary 2.1. *Let H_1 be an γ_1 regular graph on n_1 vertices and m_1 edges, and H_2 an arbitrary graph on n_2 vertices. Then the adjacency spectrum of $H_1 \vee_{\boxdot} H_2$ consist of:*

1. *-2, repeated $m_1 - n_1$ times.*
2. *$\theta_i(H_2)$ for $i = 1,2,3,\ldots,n_2$.*
3. *two roots of the equation $x^2 - x(2\gamma_1 - 2 + n_1\Gamma_{A(H_2)}(x)) - 2\gamma_1 + (2\gamma_1 - 2)\Gamma_{A(H_2)}(x) = 0$.*
4. *the remaining $2n_1 - 2$ roots are obtained from the equation $x^2 - x(\gamma_1 + \theta_i(H_1) - 2) - \gamma_1 - \theta_i(H_1) = 0$ where $i = 1,2,3,\ldots,n_1 - 1$*

Corollary 2.2. *Let H_i be an γ_i regular graph on n_i vertices and m_i edges for i=1,2. Then the characteristic polynomial of $H_1 \vee_{\boxdot} H_2$ is*

$$
\begin{aligned}
\phi(H_1 \vee_{\boxdot} H_2; x) = (x+2)^{m_1-n_1} \cdot (x^3 - x^2(2\gamma_1 + \gamma_2 - 2) - x(2\gamma_1\gamma_2 - 2\gamma_2 - 2\gamma_1 - n_1 n_2) + 2\gamma_1\gamma_2 + 2\gamma_1 n_2 - 2n_2) \\
\pi_{i=1}^{n_2-1}(x - \theta_i(G_2)) \cdot \Pi_{i=1}^{n_1-1}(x^2 - x(\gamma_1 + \theta_i(H_1) - 2) - \gamma_1 - \theta_i(H_1))
\end{aligned}
$$

2.2 The Laplacian Spectra of $H_1 \vee_{\boxdot} H_2$

Let H_i be a graph on n_i vertices and m_i edges. Then the Laplacian matrix of $H_1 \vee_{\boxdot} H_2$ can be written as:

$$
L(H_1 \vee_{\boxdot} H_2) = \begin{bmatrix} (r_1+n_2)I_{n_1} & -R_1 & -J_{n_1\times n_2} \\ -R_1^T & 2r_1 I_{m_1} - A(l(H_1)) & 0 \\ -J_{n_2\times n_1} & 0 & n_1 I_{n_2} + L(H_2) \end{bmatrix}
$$

Theorem 2.2. *Let H_1 be an γ_1 regular graph on n_1 vertices and m_1 edges, and H_2 an arbitrary graph on n_2 vertices. Then the Laplacian characteristic polynomial of $H_1 \vee_{\boxdot} H_2$ is*

$$
\begin{aligned}
\phi(L(H_1 \vee_{\boxdot} H_2; x)) = x \cdot (x - 2\gamma_1 - 2)^{m_1-n_1} \cdot \left(x^2 - x\left(2n_1 + n_2 + 2\right) + 2n_1 + 2n_2 - {n_1}^2\right) \cdot \\
\Pi_{i=1}^{n_2-1}\left(x - n_1 - \mu_i(H_2)\right) \cdot \\
\Pi_{i=1}^{n_1-1}\left(x^2 - x\left(2\gamma_1 + n_2 + \theta_i(H_1) + 2\right) + {\gamma_1}^2 + \gamma_1 n_2 + \gamma_1\theta_i(H_1) + n_2\theta_i(H_1) + \gamma_1 + 2n_2 - \theta_i(H_1)\right)
\end{aligned}
$$

Proof. The Characteristic polynomial of the Laplacian matrix of $H_1 \vee_{\boxdot} H_2$ is given by

$$\begin{aligned} det\,(L(H_1 \vee_{\boxdot} H_2)) &= \begin{vmatrix} (x-\gamma_1-n_2)I_{n_1} & R_1 & J_{n_1\times n_2} \\ R_1^T & (x-2\gamma_1)I_{m_1}+A(l(H_1)) & 0 \\ J_{n_2\times n_1} & 0 & (x-n_1)I_{n_2}-L(H_2) \end{vmatrix} \\ &= det((x-n_1)I_{n_2}-L(H_2))\cdot detS \\ &= \Pi_{i=1}^{n_2}\,(x-n_1-\mu_i(H_2))\cdot detS \end{aligned}$$

where

$$S = \begin{pmatrix} (x-\gamma_1-n_2)I_{n_1}-\Gamma_{L(H_2)}(x-n_1)J_{n_1\times n_1} & R_1 \\ R_1^T & (x-2\gamma_1)I_{m_1}+A(l(H_1)) \end{pmatrix}$$

$$\begin{aligned} & det(S) \\ &= det\left((x-\gamma_1-n_2)I_{n_1}-\Gamma_{L(H_2)}(x-n_1)J_{n_1\times n_1}\right)\cdot \\ &\quad det\left((x-2\gamma_1)I_{m_1}+A(l(H_1))-R_1^T((x-\gamma_1-n_2)I_{n_1}-\Gamma_{L(H_2)}(x-n_1)J_{n_1\times n_1})^{-1}R_1\right) \\ &= (x-\gamma_1-n_2)^{n_1}\cdot\left(1-\Gamma_{L(H_2)}(x-n_1)\frac{n_1}{x-\gamma_1-n_2}\right)\cdot \\ &\quad det\left(xI_{m_1}-2\gamma_1I_{m_1}+A(l(H_1))-\frac{2I_{m_1}+A(l(H_1))}{x-\gamma_1-n_2}+\frac{4\Gamma_{L(H_2)}(x-n_1)}{(x-\gamma_1-n_2)(x-\gamma_1-n_2-n_1\Gamma_{L(H_2)}(x-n_1))}J_{m_1\times m_1})\right) \\ &= (x-2\gamma_1-2)^{m_1-n_1}(x-\gamma_1-n_2)^{-1}\cdot \\ &\quad \Pi_{i=1}^{n_1-1}\left(x^2-x(2\gamma_1+n_2+\theta_i(H_1)+2)+{\gamma_1}^2+\gamma_1n_2+\gamma_1\theta_i(H_1)+\gamma_1+2n_2+n_2\theta_i(H_1)-\theta_i(H_1)\right)\cdot \\ &\quad x^3-x^2(2n_2+2\gamma_1+n_1\Gamma_{L(H_2)}(x-n_1)+2)+ \\ &\quad x(n_1n_2\Gamma_{L(H_2)}(x-n_1)+n_1\gamma_1\Gamma_{L(H_2)}(x-n_1)+2n_1+\Gamma_{L(H_2)}(x-n_1)+4n_2+{\gamma_1}^2\gamma_1) \\ &-2n_1n_2\Gamma_{L(H_2)}(x-n_1)-2{n_2}^2-2n_2\gamma_1-2n_1\gamma_1\Gamma_{L(H_2)}(x-n_1) \\ &= (x-2\gamma_1-2)^{m_1-n_1}(x-\gamma_1-n_2)^{-1}\cdot \\ &\quad (x-\gamma_1-n_2)\left(x^2-x(2+n_2+n_1+n_1\Gamma_{L(H_2)}(x-n_1))+2n_1\Gamma_{L(H_2)}(x-n_1)+2n_2\right)\cdot \\ &\quad \Pi_{i=1}^{n_1-1}\left(x^2-x(2\gamma_1+n_2+\theta_i(H_1)+2)+{\gamma_1}^2+\gamma_1n_2+\gamma_1\theta_i(H_1)+n_2\theta_i(H_1)+\gamma_1+2n_2-\theta_i(H_1)\right) \end{aligned}$$

$$\begin{aligned} \phi(L(H_1\vee_{\boxdot}H_2);x) &= (x-2\gamma_1-2)^{m_1-n_1}\cdot\Pi_{i=1}^{n_2}\,(x-n_1-\mu_i(H_2))\cdot \\ &\quad \left(x^2-x\left(2+n_2+n_1+n_1\Gamma_{L(H_2)}(x-n_1)\right)+2n_2+2n_1\Gamma_{L(H_2)}(x-n_1)\right)\cdot \\ &\quad \Pi_{i=1}^{n_1-1}\left(x^2-x\,(2\gamma_1+n_2+\theta_i(H_1)+2)+{\gamma_1}^2+\gamma_1n_2+\gamma_1\theta_i(H_1)+n_2\theta_i(H_1)+\gamma_1+2n_2-\theta_i(H_1)\right) \end{aligned}$$

For any graph H, $\mu_1(H)=0$ and $\Gamma_{L(H)}(x)=\frac{n}{x}$

Thus the characteristic polynomial of the Laplacian matrix of $H_1 \vee_{\boxdot} H_2$ is

$$\begin{aligned} \phi(L(H_1\vee_{\boxdot}H_2;x)) &= x\cdot(x-2\gamma_1-2)^{m_1-n_1}\cdot\left(x^2-x\,(2n_1+n_2+2)+2n_1+2n_2-{n_1}^2\right)\cdot \\ &\quad \Pi_{i=1}^{n_2-1}\,(x-n_1-\mu_i(H_2))\cdot \\ &\quad \Pi_{i=1}^{n_1-1}\left(x^2-x\,(2\gamma_1+n_2+\theta_i(H_1)+2)+{\gamma_1}^2+\gamma_1n_2+\gamma_1\theta_i(H_1)+n_2\theta_i(H_1)+\gamma_1+2n_2-\theta_i(H_1)\right) \end{aligned}$$

□

Corollary 2.3. *Let H_i be an γ_i regular graph on n_i vertices and m_i edges for i=1,2 and let $t(H)$ denote the number of spanning tree of H. Then for the graph $H_1 \vee_{\boxdot} H_2$*

$$t(H_1\vee_{\boxdot}H_2;x) = \frac{(2\gamma_1+2)^{m_1-n_1}\left(2n_1+2n_2-{n_1}^2\right)\Pi_{i=1}^{n_2-1}\,(n_1+\mu_i(H_2))\,\Pi_{i=1}^{n_1-1}\left({\gamma_1}^2+\gamma_1n_2+\gamma_1\theta_i(H_1)+n_2\theta_i(H_1)+\gamma_1+2n_2-\theta_i(H_1)\right)}{n_1+n_2+m_1}$$

2.3 The Signless Laplacian Spectra of $H_1 \vee_{\boxdot} H_2$

Let H_i be a graph on n_i vertices and m_i edges. Then the Signless Laplacian matrix of $H_1 \vee_{\boxdot} H_2$ can be written as:

$$Q(H_1\vee_{\boxdot}H_2) = \begin{bmatrix} (r_1+n_2)I_{n_1} & R_1 & J_{n_1\times n_2} \\ R_1^T & 2r_1I_{m_1}+A(l(H_1)) & 0 \\ J_{n_2\times n_1} & 0 & n_1I_{n_2}+Q(H_2) \end{bmatrix}$$

Theorem 2.3. *Let H_1 be an γ_1 regular graph on n_1 vertices and m_1 edges, and H_2 an arbitrary graph on n_2 vertices. Then the characteristic polynomial of the Signless Laplacian matrix of $H_1 \vee_{\boxdot} H_2$ is*

$$\phi(Q(H_1 \vee_{\boxdot} H_2); x)$$
$$= (x - 2\gamma_1 - 2)^{m_1-n_1} \cdot \Pi_{i=1}^{n_2} (x - n_1 - \eta_i(H_2)) \cdot$$
$$\left(x^2 - x\left(2 - n_2 - 5\gamma_1 + n_1\Gamma_{Q(H_2)}(x - n_1)\right) + 4\gamma_1{}^2 + 4\gamma_1 n_2 - 2n_2 + 4\gamma_1 n_2 \Gamma_{Q(H_2)}(x - n_1) - 2n_1\Gamma_{Q(H_2)}(x - n_1)\right) \cdot$$
$$\Pi_{i=1}^{n_1-1} \left(x^2 - x(4\gamma_1 + n_2 + \theta_i(H_1) - 2) + 3\gamma_1{}^2 + 3\gamma_1 n_2 - 3\gamma_1 - 2n_2 + \gamma_1\theta_i(H_1) + n_2\theta_i(H_1) - \theta_i(H_1)\right)$$

Proof. The proof of the theorem is similar to that of Theorem 2.2 □

3 Spectra of Q-edge join of graphs

3.1 The Adjacency Spectra of $H_1 \vee_{\boxminus} H_2$

Let H_i be a graph on n_i vertices and m_i edges. Then the adjacency matrix of $H_1 \vee_{\boxminus} H_2$ can be written as:

$$A(H_1 \vee_{\boxminus} H_2) = \begin{bmatrix} 0 & R_1 & 0 \\ R_1^T & A(l(H_1)) & J_{m_1\times n_2} \\ 0 & J_{n_2\times m_1} & A(H_2) \end{bmatrix}$$

Theorem 3.1. *Let H_1 be an γ_1 regular graph on n_1 vertices and m_1 edges, and H_2 an arbitrary graph on n_2 vertices. Then the characteristic polynomial of adjacency matrix of $H_1 \vee_{\boxminus} H_2$ is*

$$\phi(H_1 \vee_{\boxminus} H_2; x) = \phi(H_2; x) \cdot (x+2)^{m_1-n_1} \left(x^2 - x(2\gamma_1 - 2 + m_1\Gamma_{(H_2)}(x)) - 2\gamma_1\right) \Pi_{i=1}^{n_1-1}(x^2 - x(\gamma_1 + \theta_i(H_1) - 2) - \gamma_1 - \theta_i(H_1))$$

Proof. The Characteristic polynomial of the adjacency matrix of $H_1 \vee_{\boxminus} H_2$ is given by

$$\begin{aligned} det A(H_1 \vee_{\boxminus} H_2) &= \begin{vmatrix} xI_{n_1} & -R_1 & 0 \\ -R_1^T & xI_{m_1} - A(l(H_1)) & -J_{m_1\times n_1} \\ 0 & -J_{n_2\times m_1} & xI_{n_2} - A(H_2) \end{vmatrix} \\ &= det(xI_{n_2} - A(H_2)) \cdot det S \\ &= \phi(H_2; x) \cdot det S \end{aligned}$$

where

$$S = \begin{pmatrix} xI_{n_1} & -R_1 \\ -R_1^T & xI_{m_1} - A(l(H_1)) - \Gamma_{A(H_2)}(x) J_{m_1\times m_1} \end{pmatrix}$$

$$\begin{aligned} det(S) &= det(xI_{n_1}) \cdot det\left(xI_{m_1} - A(l(H_1)) - \Gamma_{A(H_2)}(x) J_{m_1\times m_1} - \frac{R_1^T R_1}{x}\right) \\ &= x^{n_1} \cdot det\left(xI_{m_1} - A(l(H_1)) - \frac{2I_{m_1} + A(l(H_1))}{x} - \Gamma_{A(H_2)}(x)) J_{m_1\times m_1}\right) \\ &= x^{n_1} \cdot (1 - \Gamma_{A(H_2)}(x)\Gamma_{A(l(H_1)) + \frac{2I_{m_1} + A(l(H_1))}{x}}(x)) \cdot det\left(xI_{m_1} - A(l(H_1)) - \frac{2I_{m_1} + A(l(H_1))}{x}\right) \end{aligned}$$

Since H_1 is an γ_1- regular graph, $\Gamma_{A(l(H_1)) + \frac{2I_{m_1} + A(l(H_1))}{x}}(x) = \frac{m_1 x}{x^2 - x(2\gamma_1 - 2) - 2\gamma_1}$

$$\begin{aligned} det S &= x^{n_1}\left(1 - \frac{m_1\Gamma_{A(H_2)}(x)}{x^2 - x(2\gamma_1 - 2) - 2\gamma_1}\right) \cdot \quad \Pi_{m_1}^{i=1}\left(x - \theta_i(L(H_1)) - \frac{2 + \theta_i(l(H_1))}{x}\right) \\ &= (x+2)^{m_1-n_1}\left(x^2 - x(2\gamma_1 - 2 + m_1\Gamma_{A(H_2)}(x)) - 2\gamma_1\right) \cdot \Pi_{n_1-1}^{i=1}\left(x^2 - x(\gamma_1 - 2 + \theta_i(H_1)) - \gamma_1 - \theta_i(H_1)\right) \end{aligned}$$

Thus the characteristic polynomial of $H_1 \vee_{\boxminus} H_2$is

$$\begin{aligned} \phi(H_1 \vee_{\boxminus} H_2; x) = \phi(H_2; x) \cdot (x+2)^{m_1-n_1} \cdot \\ \left(x^2 - x(2\gamma_1 - 2 + m_1\Gamma_{A(H_2)}(x)) - 2\gamma_1\right) \cdot \\ \Pi_{i=1}^{n_1-1}\left(x^2 - x(\gamma_1 - 2 + \theta_i(H_1)) - \gamma_1 - \theta_i(H_1)\right) \end{aligned}$$

Since H_1 is a regular graph with regularity γ_1, in the last step we use the fact that $\gamma_1 n_1 = 2m_1$ and $\theta_1(H_1) = \gamma_1$ □

Corollary 3.1. *Let H_1 be an γ_1 regular graph on n_1 vertices and m_1 edges, and H_2 an arbitrary graph on n_2 vertices. Then the adjacency spectrum of $H_1 \vee_{\boxminus} H_2$ consist of:*

1. -2, repeated $m_1 - n_1$ times.
2. $\theta_i(H_2)$ for $i = 1, 2, 3, \ldots, n_2$.
3. two roots of the equation $x^2 - x(2\gamma_1 - 2 + m_1\Gamma_{A(H_2)}(x)) - 2\gamma_1$.
4. the remaining $2n_1 - 2$ roots are obtained from the equation $\left(x^2 - x\left(\gamma_1 - 2 + \theta_i(H_1)\right) - \gamma_1 - \theta_i(H_1)\right)$ where $i = 1, 2, 3, \ldots, n_1 - 1$

Corollary 3.2. *Let H_i be an γ_i regular graph on n_i vertices and m_i edges for i=1,2. Then the characteristic polynomial of $H_1 \vee_{\boxminus} H_2$ is*

$$\phi(H_1 \vee_{\boxminus} H_2; x) = (x+2)^{m_1 - n_1} \cdot (x^3 - x^2(2\gamma_1 + \gamma_2 - 2) - x(2\gamma_1\gamma_2 - 2\gamma_2 - 2\gamma_1 - m_1 n_2) + 2\gamma_1\gamma_2)$$
$$\pi_{i=1}^{n_2 - 1}(x - \theta_i(G_2)) \cdot \Pi_{i=1}^{n_1 - 1}(x^2 - x(\gamma_1 + \theta_i(H_1) - 2) - \gamma_1 - \theta_i(H_1))$$

3.2 The Laplacian Spectra of $H_1 \vee_{\boxminus} H_2$

Let H_i be a graph on n_i vertices and m_i edges. Then the Laplacian matrix of $H_1 \vee_{\boxminus} H_2$ can be written as:

$$L(H_1 \vee_{\boxminus} H_2) = \begin{bmatrix} r_1 I_{n_1} & -R_1 & 0 \\ -R_1^T & (2r_1 + n_2)I_{m_1} - A(l(H_1)) & -J_{m_1 \times n_2} \\ 0 & -J_{n_2 \times m_1} & m_1 I_{n_2} + L(H_2) \end{bmatrix}$$

Theorem 3.2. *Let H_1 be an γ_1 regular graph on n_1 vertices and m_1 edges, and H_2 an arbitrary graph on n_2 vertices. Then the characteristic polynomial Laplacian matrix of $H_1 \vee_{\boxminus} H_2$ is*

$$\phi(L(H_1 \vee_{\boxminus} H_2); x) = (x - 2\gamma_1 - n_2 - 2)^{m_1 - n_1} \cdot \Pi_{i=1}^{n_2}\left(x - m_1 - \mu_i(H_2)\right) \cdot$$
$$\left(x^2 - x\left(\gamma_1 + n_2 + 2\right) + \gamma_1 n_2 - m_1(x - \gamma_1)\Gamma_{L(H_2)}(x - m_1)\right) \cdot$$
$$\Pi_{i=1}^{n_1 - 1}\left(x^2 - x\left(2\gamma_1 + n_2 - \theta_i(H_1) + 2\right) + {\gamma_1}^2 + \gamma_1 n_2 + \gamma_1 - \gamma_1\theta_i(H_1) - \theta_i(H_1)\right)$$

Proof. The Characteristic polynomial of the Laplacian matrix of $H_1 \vee_{\boxminus} H_2$ is given by

$$det(L(H_1 \vee_{\boxminus} H_2)) = \begin{vmatrix} (x - \gamma_1)I_{n_1} & R_1 & 0 \\ R_1^T & (x - 2\gamma_1 + n_2)I_{m_1} + A(l(H_1)) & J_{m_1 \times n_2} \\ 0 & J_{n_2 \times m_1} & (x - m_1)I_{n_2} - L(H_2) \end{vmatrix}$$
$$= det\left((x - m_1)I_{n_2} - L(H_2)\right) \cdot det(S)$$
$$= \Pi_{i=1}^{n_2}\left(x - m_1 - \mu_i(H_2)\right) \cdot det(S)$$

where

$$S = \begin{pmatrix} (x - \gamma_1)I_{n_1} & R_1 \\ R_1^T & (x - 2\gamma_1 - n_2)I_{m_1} + A(l(H_1)) - \Gamma_{L(H_2)}(x - m_1)J_{m_1 \times m_1} \end{pmatrix}$$

$$det(S) = det\left((x - \gamma_1)I_{n_1}\right) \cdot det\left((x - 2\gamma_1 - n_2)I_{m_1} + A(l(H_1)) - \Gamma_{L(H_2)}(x - m_1)J_{m_1 \times m_1} - \frac{R_1^T R_1}{x - \gamma_1}\right)$$
$$= (x - \gamma_1)^{n_1} \cdot det\left((x - 2\gamma_1 - n_2)I_{m_1} + A(l(H_1)) - \Gamma_{L(H_2)}(x - m_1)J_{m_1 \times m_1} - \frac{2I_{m_1} + A(l(H_1))}{(x - \gamma_1)})\right)$$
$$= (x - \gamma_1)^{n_1} \cdot \left(1 - \frac{m_1(x - \gamma_1)\Gamma_{L(H_2)}(x - m_1)}{x^2 - x(\gamma_1 + n_2 + 2) + n_2\gamma_1}\right)(x - 2\gamma_1 - n_2 - 2)^{m_1 - n_1} \cdot$$
$$\Pi_{n_1}^{i=1}\left(x^2 - x\left(2\gamma_1 + n_2 - \theta_i(H_1) + 2\right) + {\gamma_1}^2 + \gamma_1 n_2 + \gamma_1 - \gamma_1\theta_i(H_1) - \theta_i(H_1)\right)(x - \gamma_1)^{-n_1}$$

Thus the characteristic polynomial of laplacian matrix of $H_1 \vee_{\boxminus} H_2$is

$$\phi(L(H_1 \vee_{\boxminus} H_2); x) = (x - 2\gamma_1 - n_2 - 2)^{m_1 - n_1} \cdot \Pi_{i=1}^{n_2}\left(x - m_1 - \mu_i(H_2)\right) \cdot$$
$$\left(x^2 - x\left(\gamma_1 + n_2 + 2\right) + \gamma_1 n_2 - m_1(x - \gamma_1)\Gamma_{L(H_2)}(x - m_1)\right) \cdot$$
$$\Pi_{i=1}^{n_1 - 1}\left(x^2 - x\left(2\gamma_1 + n_2 - \theta_i(H_1) + 2\right) + {\gamma_1}^2 + \gamma_1 n_2 + \gamma_1 - \gamma_1\theta_i(H_1) - \theta_i(H_1)\right)$$

Since H_1 is a regular graph with regularity γ_1, in the last step we use the fact that $\gamma_1 n_1 = 2m_1$ and $\theta_1(H_1) = \gamma_1$ □

Corollary 3.3. *Let H_i be an γ_i regular graph on n_i vertices and m_i edges for i=1,2 and let $t(H)$ denote the number of spanning tree of H. Then for the graph $H_1 \vee_{\boxdot} H_2$*

$t(H_1 \vee_{\boxminus} H_2; x) =$

$$\frac{(2\gamma_1 + n_2 + 2)^{m_1 - n_1}\left(\gamma_1 m_1 + n_2 m_1 + n_2\gamma_1 - m_1 n_2 + 2m_1\right)\Pi_{i=1}^{n_2 - 1}\left(m_1 + \mu_i(H_2)\right)\Pi_{i=1}^{n_1 - 1}\left({\gamma_1}^2 + \gamma_1 n_2 - \gamma_1\theta_i(H_1) + \gamma_1 - \theta_i(H_1)\right)}{n_1 + n_2 + m_1}$$

3.3 The Signless Laplacian Spectra of $H_1 \vee_{\boxminus} H_2$

Let H_i be a graph on n_i vertices and m_i edges. Then the Signless Laplacian matrix of $H_1 \vee_{\boxminus} H_2$ can be written as:

$$Q(H_1 \vee_{\boxminus} H_2) = \begin{bmatrix} r_1 I_{n_1} & R_1 & 0 \\ R_1^T & (2r_1+n_2)I_{m_1} + A(l(H_1)) & -J_{m_1 \times n_2} \\ 0 & -J_{n_2 \times m_1} & m_1 I_{n_2} + Q(H_2) \end{bmatrix}$$

Theorem 3.3. *Let H_1 be an γ_1 regular graph on n_1 vertices and m_1 edges, and H_2 an arbitrary graph on n_2 vertices. Then the characteristic polynomial of SignlessLaplacian of $H_1 \vee_{\boxminus} H_2$ is*

$$\phi(Q(H_1 \vee_{\boxminus} H_2); x) = (x - 2\gamma_1 - n_2 + 2)^{m_1 - n_1} \cdot \Pi_{i=1}^{n_2} (x - m_1 - \eta_i(H_2)) \cdot \left(x^2 - x(5\gamma_1 + n_2 - 2) + 4{\gamma_1}^2 + n_2\gamma_1 - 4\gamma_1 - m_1(x - \gamma_1)\Gamma_{Q(H_2)}(x - m_1)\right) \cdot \Pi_{i=1}^{n_1-1} \left(x^2 - x(4\gamma_1 + n_2 + \theta_i(H_1) - 2) + 3{\gamma_1}^2 - 3\gamma_1 + \gamma_1 n_2 + \gamma_1\theta_i(H_1) - \theta_i(H_1)\right)$$

Proof. The proof of the theorem is similar to that of Theorem 3.2 □

4 Construction of cospectral graphs

Infinitely many pairs of co spectral graphs are constructed using different graph operations. In this paper as an application we constructed infinitely many pairs of co spectral graphs using Q-vertex join and Q-edge join of graphs.From the theorems of Section 2 and 3 we get the following results.

Theorem 4.1. *If H_1 and H_2 are $A-$cosepctral regular graph and H is any graph, then*

1. *$H_1 \vee_{\boxdot} H$ and $H_2 \vee_{\boxdot} H$ are $A-$cospectral.*
2. *$H_1 \vee_{\boxminus} H$ and $H_2 \vee_{\boxminus} H$ are $A-$cospectral.*
3. *$H \vee_{\boxdot} H_1$ and $H \vee_{\boxdot} H_2$ are $A-$cospectral.*
4. *$H \vee_{\boxminus} H_1$ and $H \vee_{\boxminus} H_2$ are $A-$cospectral.*
5. *$H_1 \vee_{\boxdot} H$ and $H_2 \vee_{\boxdot} H$ are $L-$cospectral.*
6. *$H_1 \vee_{\boxminus} H$ and $H_2 \vee_{\boxminus} H$ are $L-$cospectral.*
7. *$H_1 \vee_{\boxdot} H$ and $H_2 \vee_{\boxdot} H$ are $Q-$cospectral.*
8. *$H_1 \vee_{\boxminus} H$ and $H_2 \vee_{\boxminus} H$ are $Q-$cospectral.*

Theorem 4.2. *If H_1 and H_2 are $A-$cosepctral regular graph and H' and H'' are $L-$cospectral then*

1. *$H_1 \vee_{\boxdot} H'$ and $H_2 \vee_{\boxdot} H$" are $L-$cospectral.*
2. *$H_1 \vee_{\boxminus} H'$ and $H_2 \vee_{\boxminus} H$" are $L-$cospectral.*

Theorem 4.3. *If H is a regular graph and H_1 and H_2 are $Q-$cosepctral graph and H' and H'' with $\Gamma_{Q(H_1)}(x) = \Gamma_{Q(H_2)}(x)$ then*

1. *$H \vee_{\boxdot} H_1$ and $H \vee_{\boxdot} H_2$ are $Q-$cospectral.*
2. *$H \vee_{\boxminus} H_1$ and $H \vee_{\boxminus} H_2$ are $Q-$cospectral.*

5 Conclusion

In this paper, we determine the spectra of Q-vertex join and Q-edge join and we construct infinitely many pairs of $A-$cospectral mates, $L-$cospectral mates, $Q-$cospectral mates as an application.

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